P68 - 69

Solutions to J11BA pps 66-85. REV

Solutions that are typeset have been check
Others may or may not have been checke

[1.1] $a_{20} = 4 + 19(3) = 61$

[1.2] $a_{25} = 7 + 24(-\frac{1}{3}) = -1$

[2.1] $a = 23,\ d = (36-23) = 7 \Rightarrow a_n = a + (n-1)7$

[2.2] $a = 2,\ d = \frac{5}{4} - 2 = -\frac{3}{4} \Rightarrow a_n = 2 + (n-1)(-\frac{3}{4})$

[3] $-37 = 8 + (n-1)(-3) \Rightarrow n = 16 \quad \therefore a_{16}$

[4] $\begin{matrix} a_3 = -4 \\ a_{10} = 38 \end{matrix} \Rightarrow \begin{bmatrix} a + 9d = 38 \\ a + 2d = -4 \end{bmatrix} \Rightarrow a_n = -16 + (n-1)6$

[5] $a = 3,\ a_{15} = 94,$

$3 + 14d = 94 \Rightarrow d = \frac{13}{2}$

$a_5 = 3 + 4(\frac{13}{2}) = 29$

$a_{10} = 3 + 9(\frac{13}{2}) = \frac{123}{2}$

P70

[6] Difference of sequential terms is $a_{n+1} - a_n$,

$[p(n+1)+q] - [pn+q] = p$

Since by hypothesis, p constant, $\{pn+q\}$ is arithmetic sequence.

[7] $a_n = a + (n-1)d_a$ and $b_n = b + (n-1)d_b$

$a_n + b_n = a + d_a n - d_a + b + d_b n - d_b$

$= (a+b) + d_a n - d_a + d_b n - d_b$

$= (a+b) + d_a(n-1) + d_b(n-1)$

$= (a+b) + (d_a + d_b)(n-1)$

$= \alpha + \beta(n-1)$ where α, β constants.

P71

[8.1] $a=7,\ l=61,\ n=10$

$$S_{10} = \frac{10(7+61)}{2} = 340$$

[8.2] $a=-10,\ d=4,\ n=13$

$$S_{13} = \frac{13\,[(-20)+(12)4]}{2} = 182$$

[8.3] $a=21,\ d=-6,\ l=-117$

$$S_n = \frac{n(a+l)}{2} = \frac{n[2a+(n-1)d]}{2}$$

$$\Rightarrow \quad -48n = \frac{n[42+(n-1)(-6)]}{2}$$

$$\Rightarrow \quad -96n = 42n - 6n^2 + 6n$$

$$\Rightarrow \quad +16n = 7n - n^2 + n$$

$$\Rightarrow \quad n^2 - 24n = 0$$

$$\Rightarrow \quad n(n-24)=0 \Rightarrow n=0 \text{ or } n=24$$

$$S_n = \frac{24(-96)}{2}$$

$$S_n = -1152$$

p71, ctd

[9.1] Prove $1+2+3+\ldots+n = \frac{n(n+1)}{2}$

Proof $S_n = 1+2+3+\ldots+n$

$S_n = n+(n-1)+(n-2)+\ldots$ [$n-(n-1)$]

$2S_n = \underbrace{(n+1)+(n+1)+(n+1)+\ldots+n+1}_{n\text{-terms}}$

$\therefore S_n = \frac{n(n+1)}{2}$

[9.2] Prove $1+3+5+\ldots+(2n-1) = n^2$

Proof

$S_n = 1 + 3 + 5 + \ldots + (2n-5)+(2n-3)+(2n-1)$

$S_n = (2n-1)+(2n-3)+\ldots+5+3+1$

$\Rightarrow 2S_n = \underbrace{2n+2n+\ldots+2n}_{n\text{ terms}}$

$= (2n)n$

$= 2n^2$

$\therefore S_n = n^2$

[10.1] $S_{100} = \frac{100(101)}{2} = 5050$

[10.2] $S_{66} = 3+6+9+12+\ldots+3n+$

$n \in Z^+,\ 3n < 200$

$n < 198$

$= \frac{66(3+198)}{2}$

$\frac{200}{3} = 66\ R\ 2$

so $3 \cdot 66 = 198$

Last mult of 3 before 200

$S_{66} = 6633$

P 71, ctd

[11] $S_n = 297$, $a = 45$, $d = -3$; $n = ?$

$$297 = \frac{n}{2}\left[90 + (n-1)(-3)\right]$$
$$= \frac{n}{2}\left[90 - 3n + 3\right]$$

$\Rightarrow$ $n = 3$ or $n = 22$

$S_3 = 128$ and $S_{22} = 297$

$\therefore n = 22$

END 3.1.1 – 3.1.2

[1.1] $a_n = ar^{n-1}$

$a_6 = 1 \cdot 2^5$

$a_6 = 32$

[1.2] $a_5 = 3\left(-\frac{1}{3}\right)^4 = 3\left(\frac{1}{81}\right) = \frac{1}{27}$

[2.1] $a = \sqrt{2}$, $r = \sqrt{2}$, $a_n = \sqrt{2}\,(\sqrt{2})^{n-1}$

[2.2] $a = 1$, $r = -3$, $a_n = 1 \cdot (-3)^{n-1}$

[2.3] $a = 1$, $r = (-1)$, $a_n = 1 \cdot (-1)^{n-1}$

[3] $a_n = \dfrac{3^{n+1}}{2^n}$

$a_1 = \dfrac{3^2}{2^1} = \dfrac{9}{2}$

$r = \dfrac{3^{n+2}}{2^{n+1}} \cdot \dfrac{2^n}{3^{n+1}} = \dfrac{3}{2}$

[4] $a_3 = 4$, $a_5 = 36$

$\left.\begin{matrix} 4 = ar^2 \\ 36 = ar^4 \end{matrix}\right\}$ $9 = r^2 \Rightarrow r = 3$ or $r = -3$

then $4 = a(3)^2$ or $4 = a(-3)^2$

so $a = \frac{4}{9}$

$\therefore a = \frac{4}{9}$, $r = 3$ or $r = -3$

[5.1] $a = 3,\ r = 2,\ n = 7$, get S_7

$$S_7 = \frac{3(1-2^n)}{1-2} = -3(1-2^n) = -3 + 3\cdot 2^n$$

$$= -3 + 3\cdot 2^7 = 381$$

[5.2] $a = 1,\ r = -3,\ n = 6$

$$S_6 = \frac{1-(-3)^6}{1+3} = \frac{1}{4}(1-(-3)^6) = -182$$

[5.3] $a = 5,\ r = \frac{1}{2},\ n = 5$

$$S_5 = \frac{5(1-(\frac{1}{2})^5)}{\frac{1}{2}} = 10(1 - \frac{1}{32}) = 10\cdot\frac{31}{32} = \frac{310}{32} = \frac{155}{16}$$

[6.1] $13, 52, 208, 832, \ldots$

$$r = \frac{52}{13} = 4,\quad a = 13$$

$$S_n = -\frac{13(1-4^n)}{3}$$

[6.2] $\sqrt{3}, -1, \frac{1}{\sqrt{3}}, -\frac{1}{3}$

$$r = -\frac{1}{\sqrt{3}},\quad a = \sqrt{3}$$

■ **p. 75**

[7] Loan. {Note: the payment is the same amount every period}. Given:

> $A =$ initial balance (aka amount borrowed, principal borrowed)
> $r =$ rate of interest per payment period
> $n =$ number of payments
> $a =$ amount of each payment

Solution

Let a_k =balance owed *after* the k^{th} payment has been made.

Then,

$A_0 = A$

$A_1 = A_0(1+r) - a = A(1+r) - a$

$A_2 = A_1(1+r) - a = [A(1+r) - a]\,(1+r) - a = A(1+r)^2 - a(1+r) - a$

$A_3 = A_2(1+r) - a = [A(1+r)^2 - a(1+r) - a]\,(r+1) - a = A(1+r)^3 - a(1+r)^2 - a(r+1) - a$

. . .

$A_k = A(1+r)^k - a(1 + (r+1) + (r+1)^2 + \ldots + (r+1)^{k-1})$

Note that $1 + (r+1) + (r+1)^2 + \ldots + (r+1)^{k-1}$ is a geometric series, first term 1, ratio $(r+1)$. So,

$A_k = A(1+r)^k - a\,\frac{1-(r+1)^k}{1-(r+1)}$

Thus, the after the n^{th} payment, the balance is

$A_n = A(1+r)^n - a\,\frac{1-(r+1)^n}{1-(r+1)}$.

But, after the last payment, the balance is also zero, so that

$A_n = A(1+r)^n - a\,\frac{1-(r+1)^n}{1-(r+1)} = 0$

$\Longrightarrow a\,\frac{1-(r+1)^n}{1-(r+1)} = A(1+r)^n$

$\Longrightarrow a = -A\,r\,\frac{(1+r)^n}{1-(r+1)^n}$

$\therefore\ a = A\,r\,\frac{(1+r)^n}{(1+r)^n-1}$

[8] This has been checked.

$$S_n = 1\cdot 2 + 4\cdot 2^2 + 7\cdot 2^3 + \dots + [1+3(n-1)]\,2^n$$

$$\tfrac{1}{2} S_n = 1 + 4\cdot 2 + 7\cdot 2^2 + 10\cdot 2^3 + \dots + [1+3(n-1)]\,2^{n-1}$$

$$\tfrac{1}{2} S_n - S_n = 1 + 3\cdot 2 + 3\cdot 2^2 + 3\cdot 2^3 + \dots + 3\cdot 2^{n-1} - [1+3(n-1)]\,2^n$$

$$= 1 + 3\Big[\underbrace{2 + 2\cdot 2 + 2\cdot 2^2 + \dots + 2\cdot 2^{n-2}}_{\substack{\text{geom series} \\ a=2 \text{ and } r=2,\ n-1 \text{ terms}}}\Big] - [1+3(n-1)]\,2^n$$

$$= 1 + 3\left[\frac{2(1-2^{n-1})}{1-2}\right] - [1+3(n-1)]\,2^n$$

$$-\tfrac{1}{2} S_n = 1 - 3\,[\,2(1-2^{n-1})\,] - [1+3(n-1)]\,2^n$$

$$\Rightarrow \quad S_n = -2 + 6[2(1-2^{n-1})] + 2\cdot 2^n\,[3n-2]$$

$$= -2 + 12 - 12\cdot 2^{n-1} + 3n\cdot 2^{n+1} - 2^{n+2}$$

$$= 10 - 12\cdot 2^{n-1} + 3n\cdot 2^{n+1} - 2^{n+2}$$

$$= 10 - 3\cdot 2^{n+1} + 3n\cdot 2^{n+1} - 2^{n+2}$$

$$= 10 + 2^{n+1}(3n-3) - 2^{n+2}$$

$$= 10 + 3(n-1)\,2^{n+1} - 2^{n+2}$$

$$= 10 + 3(n-1)\,2^{n+1} - 2\cdot 2^{n+1}$$

$$= 10 + [3(n-1)-2]\,2^{n+1}$$

$$= 10 + [3n-3-2]\,2^{n+1}$$

$$= 10 + [3n-5]\,2^{n+1}$$

$$\therefore \quad S_n = 10 + [3n-5]\cdot 2^{n+1}$$

NOTE: when $n = 10$, $S_{10} = 51210$. If you compute S_{10} with your result and $S_{10} \neq 51210$, then your result is not EQUIV TO this one.

[2] ctd

$$(n+1)^4 - 1^4 = 4[1^3+2^3+\ldots+n^3] + 6[1^2+2^2+3^2+\ldots+n^2] + 4[1+2+3+\ldots+n] + n$$

$$4[1^3+2^3+\ldots+n^3] = (n+1)^4 - 6[1^2+2^2+3^2+\ldots+n^2] - 4[1+2+3+\ldots+n] - (n+1)$$

$$= (n+1)^4 - 6\left[\frac{n(n+1)(2n+1)}{6}\right] - 4\left[\frac{n(n+1)}{2}\right] - (n+1)$$

$$= (n+1)\left[(n+1)^3 - n(2n+1) - 2n - 1\right]$$

$$= (n+1)(n^3+3n^2+3n+1 - 2n^2 - n - 2n - 1]$$

$$= (n+1)(n^3+n^2)$$

$$= (n+1)^2 n^2$$

$$\therefore [1^3+2^3+3^3+\ldots+n^3] = \left[\frac{n(n+1)}{2}\right]^2$$

<u>Then</u>

$$[6^3+7^3+8^3+\ldots+13^3] = \left(\frac{13(14)}{2}\right)^2 - \left(\frac{5(6)}{2}\right)^2$$

$$= 8056$$

p 81

[3.1] $\sum_{i=1}^{n} i^3$

[3.2] $\sum_{k=1}^{10} [5+4(k-1)]$

$5+4k-4 = 4k+1$

$4k+1=41 \Rightarrow 4k=4 \Rightarrow k=10$

[4.1] $\sum_{k=1}^{8} (3k+1) = 4+7+10+13+16+19+22+25$

[4.2] $\sum_{i=0}^{4} \left(\frac{1}{2}\right)^i = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$

[4.3] $\sum_{j=1}^{n} j(j+1) = 2+6+12+\ldots+n(n+1)$

p 81

[5.1]
$$\sum_{k=1}^{n} (5k+1) = 5\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$
$$= 5\left[\frac{n(n+1)}{2}\right]+n$$
$$= \frac{5n(n+1)}{2}+\frac{2n}{2}$$
$$= \frac{5n^2+5n+2n}{2}$$
$$= \frac{1}{2}\left[5n^2+7n\right]$$
$$= \frac{1}{2}n\left[5n+7\right]$$

P 81

[5.2] $\sum_{i=1}^{n-1} (i+1)(i-2)$

$$= \sum_{i=1}^{n-1} i^2 - i - 2$$

$$= \sum_{i=1}^{n-1} i^2 - \sum_{i=1}^{n-1} i - \sum_{i=1}^{n-1} 2$$

$$= \frac{(n-1)(n)(2n-1)}{6} - \frac{(n-1)(n)}{2} - 2(n-1)$$

$$= \frac{n-1}{6}\left[n(2n-1) - 3n - 12\right]$$

$$= \frac{n-1}{6}\left[2n^2 - n - 3n - 12\right]$$

$$= \frac{n-1}{6}\left(2n^2 - 4n - 12\right)$$

$$= \frac{n-1}{3}\left(n^2 - 2n - 6\right)$$ 4 - 24

$$= \frac{(n-1)(n^2-2n-6)}{3}$$ CHECKED.

P 81

[6.1] $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + n(n+1)$

$$= \sum_{i=1}^{n} i(i+1) = \sum_{i=1}^{n} i^2 + i = \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n+1}{6}\left[n(2n+1) + 3n\right]$$

$$= \frac{n+1}{6}\left[2n^2 + n + 3n\right]$$

$$= \frac{n+1}{3}\left[n^2 + 2n\right]$$

$$= \frac{n(n+1)(n+2)}{3}$$

[6.2] $= 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \dots + n(n+1)(n+2)$

$$= \sum_{k=1}^{n} k(k+1)(k+2) = \sum_{k=1}^{n} k^3 + 3k^2 + 2k$$

$$= \frac{n^2(n+1)^2}{4} + \cancel{3}\left[\frac{n(n+1)(2n+1)}{\cancel{6}\,2}\right] + \cancel{2}\left[\frac{n(n+1)}{\cancel{2}}\right]$$

$$= \frac{1}{4} n(n+1)\left[n(n+1) + 2(2n+1) + 4\right]$$

$$= \frac{1}{4} n(n+1)\left[n^2 + n + 4n + 2 + 4\right]$$

$$= \frac{1}{4} n(n+1)\left[n^2 + 5n + 6\right]$$

$$= \frac{1}{4} n(n+1)(n+2)(n+3)$$

[7.1]

$1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \ldots + n(n+2)$

$$\sum_{k=1}^{n} k(k+2) = \sum_{k=1}^{n} k^2 + 2k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2k$$

$$= \frac{n(n+1)(2n+1)}{6} + n(n+1)$$

$$= \frac{1}{6}\left[n(n+1)(2n+1) + 6n(n+1)\right]$$

$$= \frac{n+1}{6}\left[2n^2 + n + 6n\right]$$

$$= \frac{n+1}{6}\left[2n^2 + 7n\right]$$

$$= \frac{n(n+1)(n+7)}{6}$$

[7.2] $1^2\cdot 2 + 2^2\cdot 5 + 3^2\cdot 8 + 4^2\cdot 11 + 5^2\cdot 14 + \ldots + n^2(2+3(n-1))$

$$= \sum_{k=1}^{n} k^2(3k-1)$$

$$= \sum_{k=1}^{n} 3k^3 - k^2 = 3\sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2$$

$$= 3\left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{3n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{12}\left[9n^2(n+1)^2 - 2n(n+1)(2n+1)\right]$$

$$= \frac{n(n+1)}{12}\left[9n(n+1) - 2(2n+1)\right]$$

NEXT $\longrightarrow$

p 81

[7.2] ctd

$$= \frac{n(n+1)}{2}\left[9n^2 + 9n - 4n - 2\right]$$

$$= \frac{n(n+1)}{2}\left[9n^2 + 5n - 2\right]$$ checked ✓ 16 4(9)(11)

P 83

[8.1] $1, 2, 5, 10, 17, 26, \ldots$

$1, 3, 5, 7, 9, \ldots = \{b_k\}$

$$a_n = a_1 + \sum_{k=1}^{n-1} b_k$$

$$= 1 + \sum_{k=1}^{n-1} [1 + 2(k-1)]$$

$$= 1 + \sum_{k=1}^{n-1} 1 + 2\sum_{k=1}^{n-1} k - 2\sum_{k=1}^{n-1} 1$$

$$= 1 + (n-1) + (n-1)n - 2(n-1)$$

$$= \cancel{1} + \cancel{n} - \cancel{1} + n^2 - \cancel{n} - 2n + 2$$

$$\boxed{a_n = n^2 - 2n + 2}$$

$$\sum_{k=1}^{n} k^2 - 2k + 2 = \frac{n(n+1)(2n+1)}{6} - n(n+1) + 2n$$

$$= \frac{n}{6}\left[(n+1)(2n+1) - 6(n+1) + 12\right]$$

$$= \frac{n}{6}\left[2n^2 + 3n + 1 - 6n - 6 + 12\right]$$

$$\boxed{\sum_{k=1}^{n} a^k = \frac{n}{6}\left[2n^2 - 3n + 7\right]}$$

P83

[8.2]

$\{a_n\} = 3, 4, 1, 10, -17, 64, \ldots$

$\{b_n\} = 1, -3, 9, -27, 81, \ldots$

$$a_n = 3 + \sum_{k=1}^{n-1} [-3]^{k-1}$$

$$= 3 + \frac{1 - [-3]^{n-1}}{1 - [-3]}$$

$$= 3 + \frac{1 - [-3]^{n-1}}{4}$$

$[-3]^{k-1}$ is geom series $a=1$, $r=-3$
So sum ~~for~~ of 1st $n-1$ terms of this series

$$\boxed{a_n = \frac{13 - [-3]^{n-1}}{4}}$$

$$\sum_{k=1}^{n} a_n = \sum_{k=1}^{n} \frac{13 - [-3]^{n-1}}{4}$$

$$= \frac{13n}{4} - \frac{1}{4}\left[\frac{1 - [-3]^{n}}{4}\right]$$

$[-3]^{n-1}$ geom series and desire sum of 1st n terms.

$$\boxed{\sum a_n = \frac{13n}{4} - \frac{1}{16}\left[1 - [-3]^{n}\right]}$$

NEED TO not get confused about index

p. 83 [9.1]

$$\sum_{k=1}^{n} \left(\frac{-1}{k\,(k+2)}\right) = \sum_{k=1}^{n} \frac{-1}{2} \left(\frac{1}{k} - \frac{1}{k+2}\right)$$

$$= \frac{-1}{2} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+2}\right)$$

$$= \frac{-1}{2} \left\{\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \ldots + \left(\frac{1}{n-5} - \frac{1}{n-3}\right) + \left(\frac{1}{n-4} - \frac{1}{n-2}\right) + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)\right\}$$

$$= \frac{-1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \left(\frac{-3\,n^2 - 5\,n}{4\,(n+1)\,(n+2)}\right)$$

p. 83 [9.2]

$$\sum_{k=1}^{n} \frac{1}{(2\,k)^2-1} = \sum_{k=1}^{n} \left(\frac{1}{2\,(2\,k-1)} - \frac{1}{2\,(2\,k+1)}\right)$$

$$= \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{2\,k-1} - \frac{1}{2\,k+1}\right)$$

$$= \frac{1}{2} \left\{ \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{2\,(n-3)-1} - \frac{1}{2\,(n-3)+1}\right) + \left(\frac{1}{2\,(n-2)-1} - \frac{1}{2\,(n-2)+1}\right) + \left(\frac{1}{2\,(n-1)-1} - \frac{1}{2\,(n-1)+1}\right) + \left(\frac{1}{2\,n-1} - \frac{1}{2\,n+1}\right) \right\}$$

$$= \frac{1}{2} \left\{ \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{2\,n-7} - \frac{1}{2\,n-5}\right) + \left(\frac{1}{2\,n-5} - \frac{1}{2\,n-3}\right) + \left(\frac{1}{2\,n-3} - \frac{1}{2\,n-1}\right) + \left(\frac{1}{2\,n-1} - \frac{1}{2\,n+1}\right) \right\}$$

$$= \frac{1}{2} \left(1 - \frac{1}{2\,n+1}\right)$$

$$= \frac{n}{2\,n+1}$$

[10]

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}$$

$$= \frac{1}{2} \left\{ \left(\frac{1}{2} - \frac{1}{6} \right) + \left(\frac{1}{6} - \frac{1}{12} \right) + \left(\frac{1}{12} - \frac{1}{20} \right) + \ldots + \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right) \right\}$$

$$= \frac{1}{2} \left(\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right)$$

$$= \left(\frac{n^2+3\,n}{4\,(n+1)(n+2)} \right)$$

[11] $S_n = n^3 - n + 2$.

$$
\begin{aligned}
a_n &= S_n - S_{n-1} \\
&= n^3 - n + 2 - [(n-1)^3 - (n-1) + 2] \\
&= n^3 - n + 2 - [n^3 - 3n^2 + 3n - 1 - n + 1 + 2] \\
&= \cancel{n^3} - \cancel{n} + \cancel{2} - \cancel{n^3} + 3n^2 - 3n + \cancel{1} + \cancel{n} - \cancel{1} - \cancel{2} \\
&= 3n^2 - 3n \\
&= 3n(n-1)
\end{aligned}
$$

[07-08-25-T11]
Answers to J11-BA page 85 Exercises

These have been checked

J11 Exercises p 85

[1]

Solution A. Since $S_3 = S_5$, $a_1 + a_2 + a_3 = a_1 + a_2 + a_3 + a_4 + a_5$. But this means $(a_4 + a_5)$ must contribute nothing to the sum. Thus, $a_4 + a_5 = 0 \Longrightarrow (5 + 3\,d) + (5 + 4\,d) = 0 \Longrightarrow 7\,d = -10 \Longrightarrow d = \frac{-10}{7}$.

Solution B. Use $S_n = \frac{n}{2}[2\,a + (n-1)\,d]$. Thus, $S_3 = \frac{3}{2}[10 + 2\,d] = \frac{5}{2}[10 + 4\,d] = S_5 \Longrightarrow d = \frac{-10}{7}$.

[2.1]

$$a_5 = a_1 + 4\,d = 108$$
$$a_{20} = a_1 + 19\,d = -237$$

Then, $a_1 = 200$, $d = -23$

[2.2]

$$S_n = \frac{n[2 \cdot 200 - 23\,(n-1)]}{2} = \frac{1}{2}\,(423\,n - 23\,n^2) = \frac{n}{2}\,(423 - 23\,n)$$

The roots of $y = \frac{x}{2}\,(423 - 23\,x)$are $x = 0$ and $x = \frac{423}{23}$. The extreme value, in this case its maximum,

occurs at $x = \frac{1}{2}\,(\frac{423}{23}) = 9\,\frac{9}{46}$. Therefore, $n = 9$. [Note: since $9\,\frac{9}{46}$ is closer to 9 than to 10, so the maxi mum of S_n is at 9, not 10.]

[3]

$$a_3 = a\,r^2 = 12$$
$$a_6 = a\,r^5 = 96$$

Solving simultaneously, $\frac{12}{r^2}\,(r^5) = 96 \Longrightarrow r^3 = 8 \Longrightarrow r = 2$. $a = \frac{12}{4} = 3$. So, $a_n = 3 \cdot 2^{n-1}$. Then,

[3.1]

$\sum_{k=1}^{n} (3 \cdot 2^{k-1})^2 = \sum_{k=1}^{n} 9 \cdot 2^{2\,(k-1)} = \sum_{k=1}^{n} 9 \cdot 4^{(k-1)}$.

This is the sum of a geometric series with $a = 9$ and $r = 4$. Thus, the sum of the first n terms is

$S_n = 9\,(\frac{4^n - 1}{4-1}) = 3\,(4^n - 1)$

[3.2]

Product of first *n* terms:

$$\prod_{i=1}^{n} (3 \times 2^{i-1})$$

$$= [(3 \times 2^0)\ (3 \times 2^1)\ (3 \times 2^2)\ (3 \times 2^3) \times \cdots \times (3 \times 2^{n-1})]$$

$$= 3^n\ (2^1 \times 2^2 \times 2^3 \times \cdots \times 2^{n-1})$$

$$= 3^n\ 2^{[1+2+3+\cdots+(n-1)]}$$

$$= 3^n\ 2^{\frac{n\,(n-1)}{2}}$$

[4.1]

$$\sum_{i=1}^{n} (1 + 3\ (n - 1))^2 = \sum_{i=1}^{n} (4 - 12\ i + 9\ i^2) = \sum_{i=1}^{n} 4 - \sum_{i=1}^{n} 12\ i + \sum_{i=1}^{n} 9\ i^2$$

$$= 4\ n - 12 \left(\frac{n\ (n + 1)}{2}\right) + 9 \left(\frac{n\ (n + 1)\ (2\ n + 1)}{6}\right)$$

$$= \frac{n}{2}\ (6\ n^2 - 3\ n - 1)$$

[4.2]

$$\sum_{i=1}^{n} (2\ i)\ (3 + (i - 1))^2 = \sum_{i=1}^{n} (8\ i + 8\ i^2 + 2\ i^3) = 2 \sum_{i=1}^{n} i^3 + 8 \sum_{i=1}^{n} i^2 + 8 \sum_{i=1}^{n} i$$

$$= 2 \left(\frac{n\ (n + 1)}{2}\right)^2 + 8 \left(\frac{n\ (n + 1)\ (2\ n + 1)}{6}\right) + 8 \left(\frac{n\ (n + 1)}{2}\right)$$

$$= \frac{n}{6}\ (32 + 51\ n + 22\ n^2 + 3\ n^3)$$

[5]

$\{a_n\} = 2, 3, 9, 18, 28, 37, 43, 44, \ldots$

$\{b_n\} = 1, 6, 9, 10, 9, 6, 1, \ldots$

$\{c_n\} = 5, 3, 1, -1, -3, -5, \ldots$

$$c_n = 5 - 2\,(n - 1)$$

$$b_n = 1 + \sum_{k=1}^{n-1} (5 - 2\,(k - 1))$$

$$= 1 + \sum_{k=1}^{n-1} (7 - 2\,k)$$

$$= (1 + 7\,(n - 1) - n\,(n - 1))$$

$$\therefore b_n = -6 + 8\,n - n^2$$

$$a_n = 2 + \sum_{k=1}^{n-1} (-6 + 8\,k - k^2)$$

$$\therefore a_n = \frac{1}{6}\,(48 - 61\,n + 27\,n^2 - 2\,n^3)$$

[6]

Let S_n be the sum of the first n terms of $\{a_n\}$. If $S_n = a\,n^2 + b\,n + c$, then is $\{a_n\}$ an arithmetic sequence?

Suppose $S_n = a\,n^2 + b\,n + c$. Then $S_{n-1} = a(\,n - 1)^2 + b(\,n - 1) + c$. Thus,

$$a_n = s_n - s_{n-1} = a\,n^2 + b\,n + c - (a\,(n - 1)^2 + b\,(n - 1) + c)$$

$$a_n = (b - a) + (2\,a)\,n$$

$\therefore$ This is an arithmetic sequence with first term $(b - a)$ and common difference $2\,a$. That is,

$$\{(b - a) + (n - 1)\,(2\,a)\}$$